Phys 235 Lecture Notes - Fractional Kinetics

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1 Review of Levy Flights and Processes

In the notes that follow we will be developing methods that can be used to deal with anomalous diffusion, the canonical example being the Levy flights of a particle undergoing superdiffusive motion. Anomalous diffusion is characterized by a diffusion processes that is non-linear in time. More specifically, anomalous diffusion is when the spatial variance has a non-linear time dependence as opposed to normal diffusion where the spatial variance has a linear time dependence. We have discussed Levy flights and processes extensively in our previous lectures, however we summarize the key results here to provide motivation for the development of techniques that deal with anomalous diffusion.

In working with Levy flights and processes it is easiest to deal with the characteristic function in order to calculate the moments of the probability distribution function (PDF). The characteristic function of the Levy distribution is

$$P_\alpha(q) = exp\big(- c|q|^\alpha \big)$$

where c is a constant and α is the Levy index that is restricted to $0 < \alpha \leq 2$. The key point here is that for $\alpha < 2$, the variance diverges which means that a standard Fokker-Planck approach to solving for the time evolution of the distribution function is not applicable. The characteristic function for a Levy process is given by

$$P_\alpha(q,t) = exp\big(- ct|q|^\alpha \big)$$

We can take the Fourier transform of the characteristic function to get the PDF

$$P_\alpha(x,t) = \int dq e^{iqx} e^{-ct|q|^\alpha}$$

In the case that $\alpha = 2$ then the PDF would reduce to a simple Gaussian and the Fokker-Planck approach would be applicable

$$P_2(x,t) = \frac{1}{\sqrt{Dt}} exp\bigg(- \frac{x^2}{Dt} \bigg)$$

In the case that $\alpha < 2$, if we consider the limit where $x \to \infty$ then the PDF is approximated by

$$P_\alpha(x,t) \sim \frac{t}{|x|^{\alpha+1}}$$

Thus we see that for the cases where $\alpha < 2$ the variance will diverge and therefore the Fokker-Planck method cannot be applied.

2 Review of Fokker-Planck Derivation

The review in the previous section shows that Levy flights and processes lead to anomalous diffusion which cannot be handled by a traditional Fokker-Planck (FP) approach. As we will see in later sections, the approaches that we will use to deal with anomalous diffusion can be thought of as modified or generalized versions of FP. Therefore we will use this section to review the usual FP approach so that it will be easier to compare and contrast with the generalized FP approach.

Let $W(x,t;x',t')$ be the probability density of a particle to be located at position x at time t if the particle was at position x' at time t' where $t' \le t$. Then we can define a Markov chain process for the particle to get from (x_1,t_1) to (x_3,t_3) by considering all of the possible intermediate steps of (x_2,t_2)

$$W(x_3,t_3;x_1,t_1) = \int dx_2 W(x_3,t_3;x_2,t_2)W(x_2,t_2;x_1,t_1) \tag{1}$$

We then make an assumption about the uniformity of time which allows us to write the probability density as

$$W(x,t;x',t') = W(x,x';t-t') = W(x,x';\Delta t)$$

where $\Delta t = t' - t$. We then assume that Δt is small and expand in powers of Δt.

$$W(x,x';t+\Delta t) = W(x,x';t) + \frac{\partial W(x,x';t)}{\partial t}\Delta t + ...$$

This expansion is valid provided that the limit definition of the partial derivative is well defined

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\big[W(x,x';t+\Delta t) - W(x,x';t)\big] = \frac{\partial W(x,x';t)}{\partial t} \tag{2}$$

To simplify things we will introduce the following notation

$$P(x,t) = W(x,x';t) \tag{3}$$

where we have neglected x' which we will justify later. If we set $t_3 = t + \Delta t$, $t_2 = t$, $x_3 = x$, $x_2 = y$ and if we neglect x_1 and t_1, which is akin to setting the initial position and time at zero, then equation (1) becomes

$$W(x,y;t+\Delta t) = \int dy W(x,y;\Delta t)W(y;t)$$

We can then substitute this result into equation (2) to get

$$
\begin{aligned}
\frac{\partial W(x,x';t)}{\partial t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int dyW(x,y;\Delta t)W(y;t) - W(x,x';t)\right]\\
\frac{\partial P(x,t)}{\partial t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int dyW(x,y;\Delta t)P(y,t) - P(x,t)\right]
\end{aligned}
\tag{4}
$$

So far we have made no assumptions about the nature of $W(x, y; \Delta t)$. It is important to note that the use of equation (3) is more than just a change of notation and that $P(x, t)$ and $W(x, y; \Delta t)$ are two distinct distribution functions. $P(x, t)$ is the distribution function that describes the long time behavior of the dynamics (i.e. $t \to \infty$). Whereas $W(x, y; \Delta t)$ describes the very short term behavior (i.e. $\Delta t \to 0$). For very long times the distribution function no longer depends on the initial condition, x', which is why it is valid to drop x' in equation (3).

Since $W(x, y; \Delta t)$ is the distribution that describes very short time scales, then we know that if the time for the transition (Δt) goes to zero, and if the the velocity is finite (a requirement for any physical system) then the particle should not move at all, in other words

$$\lim_{\Delta t\to 0} W(x,y;\Delta t) = \delta(x-y)$$

We can then assume that Δt is very small (not quite 0) and expand the delta function

$$W(x,y;\Delta t) = \delta(x-y) + A(y,\Delta t)\delta'(x-y) + \frac{1}{2}B(y,\Delta t)\delta''(x-y) \tag{5}$$

where we have ignored higher order terms and $A(y, \Delta t)$ and $B(y, \Delta t)$ are function that have yet to be determined. We can find express $A(y, \Delta t)$ and $B(y, \Delta t)$ as moments of W by utilizing the fact that W, being a transfer probability must satisfy two normalization requirements

$$\int W(x,y;\Delta t)dx = 1 \quad and \quad \int W(x,y;\Delta t)dy = 1$$

To find $A(x, \Delta t)$ we multiply equation (5) by $(x - y)$ and integrate over x

$$
\begin{aligned}
\int dx(x-y)W(x,y;\Delta t) = \int dx(x-y)\delta(x-y) &+ \int dx(x-y)A(y,\Delta t)\delta'(x-y)\\
&+ \int dx(x-y)\frac{1}{2}B(y,\Delta t)\delta''(x-y)
\end{aligned}
\tag{6}
$$

We then need to make use of the following integral identities involving the delta function (we will also need these integrals later in this paper)

$$
\begin{aligned}
\int g(x)\delta'(x)dx &= -\int \frac{\partial}{\partial x}\left[g(x)\right]\delta(x)dx\\
\int g(x)\delta^{(n)}(x)dx &= (-1)^n \int \frac{\partial^n}{\partial x^n}\left[g(x)\right]\delta(x)dx
\end{aligned}
\tag{7}
$$

Using the identities from equation (7) in equation (6) gives us

$$\int dx(x-y)W(x,y;\Delta t) = -A(y;\Delta t)$$

$$A(y;\Delta t) = \int dx(y-x)W(x,y;\Delta t) = \langle\langle \Delta y\rangle\rangle \tag{8}$$

Similarly, if we multiply equation (5) by $(x-y)^2 = (y-x)^2$ and then integrate over x and use the identities in equation (7) to simplify, we find an equation for $B(y;\Delta t)$

$$\int dx(y-x)^2 W(x,y;\Delta t) = (-1)^2 \times \frac{1}{2} \times 2 \times B(y;\Delta t)$$

$$B(y;\Delta t) = \int dx(y-x)^2 W(x,y;\Delta t) = \langle\langle (\Delta y)^2\rangle\rangle \tag{9}$$

Thus the functions $A(y;\Delta t)$ and $B(y;\Delta t)$ can be very conveniently expressed as moments of W which is useful because the moments correspond to physical macroscopic quantities of the system.

If we now integrate equation (5) over y instead of x as we have done previously then we get a new relation between $A(y;\Delta t)$ and $B(y;\Delta t)$

$$\int dyW(x,y;\Delta t) = \int dy\delta(x-y) + \int dyA(y,\Delta t)\delta'(x-y) + \int dy\frac{1}{2}B(y,\Delta t)\delta''(x-y)$$

$$1 = 1 + \int dy\frac{\partial A(y;\Delta t)}{\partial y}\delta(x-y) + \int dy\frac{1}{2}\frac{\partial^2 B(y,\Delta t)}{\partial^2 y}\delta(x-y)$$

$$\implies A(y;\Delta t) = \frac{1}{2}\frac{\partial B(y;\Delta t)}{\partial y} \tag{10}$$

Or we can express this equation in terms of the moments of W

$$\langle\langle \Delta y\rangle\rangle = \frac{1}{2}\frac{\partial}{\partial y}\langle\langle (\Delta y)^2\rangle\rangle \tag{11}$$

This equation is equivalent to the statement of Louisville's theorem which along with Hamilton's equations was the method originally used to derive the FK equation by L.D. Landau. Equations (10) and (11) are equivalent to the statement of microscopic reversibility also known as the detail balance principle. The final assumption we will need is a set of limit equations collectively known as the Kolmogorov conditions

$$\begin{aligned}
&\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\langle \Delta x\rangle\rangle = \mathcal{A}(x)\\
&\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\langle (\Delta x)^2\rangle\rangle = \mathcal{B}(x)\\
&\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\langle (\Delta x)^m\rangle\rangle = 0, \quad (m>2)
\end{aligned} \tag{12}$$

Physically, we can think of $\mathcal{B}(x)$ as a diffusion coefficient, that is, $\mathcal{B}(x) = \mathcal{D}(x)$ where $\mathcal{D}(x)$ is the diffusion coefficient. In that case, from equation (11) we have that

$$\mathcal{A}(x) = \frac{1}{2}\frac{\partial}{\partial x}\mathcal{D}(x)$$

Thus we can think of $A(x)$ as the convective component of the particle movement, i.e, the particle velocity. However we will continue to use $\mathcal{B}(x)$ for now to keep the notation more general. We can now derive the Fokker-Planck equation by substituting equation (5) into equation (4) to get

$$\frac{\partial P(x,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Bigg[\int dy \Big[\delta(x-y) + A(y,\Delta t)\delta'(x-y) + \frac{1}{2}B(y,\Delta t)\delta''(x-y)\Big]P(y;t) \\ -P(x,t)\Bigg]$$

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Bigg[P(x,t) - \int dy \frac{\partial}{\partial y}\Big\{A(y,\Delta t)P(y,t)\Big\}\delta(x-y) + \\ &\quad \frac{1}{2}\int dy \frac{\partial^2}{\partial y^2}\Big\{B(y,\Delta t)P(y,t)\Big\}\delta(x-y) - P(x;t)\Bigg] \\ \frac{\partial P(x,t)}{\partial t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Bigg[-\frac{\partial}{\partial x}\Big\{A(x,\Delta t)P(x,t)\Big\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big\{B(x,\Delta t)P(x,t)\Big\}\Bigg]\end{aligned}$$

Using the Kolmogorov conditions from equation (12) we finally have

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\Big\{\mathcal{A}(x)P(x,t)\Big\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big\{\mathcal{B}(x)P(x,t)\Big\}\Big] \tag{13}$$

which is the Fokker-Planck equation. We can use equation (10) to simply the FP equation

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\Big\{\frac{1}{2}\frac{\partial \mathcal{B}(x)}{\partial x}P(x,t)\Big\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big\{\mathcal{B}(x)P(x,t)\Big\} \\ \frac{\partial P(x,t)}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial x}\Big\{\mathcal{B}\frac{\partial P(x,t)}{\partial x}\Big\} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big\{\mathcal{B}(x)P(x,t)\Big\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big\{\mathcal{B}(x)P(x,t)\Big\} \\ \frac{\partial P(x,t)}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial x}\Big\{\mathcal{D}\frac{\partial P(x,t)}{\partial x}\Big\} \qquad (14)\end{aligned}$$

where we have replaced $\mathcal{B}$ with the more familiar diffusion coefficient, $\mathcal{D}$, that it,

$$\mathcal{D} = \mathcal{B} = \lim_{\Delta t \to 0} \frac{\langle\langle(\Delta x)^2\rangle\rangle}{\Delta t}$$

An important solution to the FP equation is when $\mathcal{D}$ is a constant. In this case, the solution to the FP equation is a Gaussian distribution of the form

$$P(x,t) = \frac{1}{\sqrt{2\pi\mathcal{D}t}} exp\Big(-\frac{x^2}{2\mathcal{D}t}\Big)$$

The odd moments of $P(x,t)$ are zero while the even moments are given by

$$\langle x^{2m} \rangle = \mathcal{D}_m t^m \quad (m = 1, 2, ...)$$

Thus we see that for $m = 1$ we have that the variance is proportional to time with the constant of proportionality equal to the diffusion coefficient which is characteristic of normal diffusion (as opposed to anomalous diffusion).

3 Strange Kinetics

As was discussed in Section 1, the Levy process naturally leads to variances of the form

$$\langle \delta x^2 \rangle \sim t^{\gamma}$$

where $\gamma \neq 1$. If we could have $\gamma = 1$ then we could simply use the Fokker-Planck approach as derived in Section 2. However, in general we will have cases where $\gamma \neq 1$ and we have to develop another approach that will account for this anomalous diffusion. There are two ways of dealing with this. The first is the Continuous Time Random Walk (CTRW) method and the second is the method of Fractional Kinetics (FK). Both methods are very similar to the FP method.

4 CTRW Model

4.1 Overview

In the CTRW model, the key idea is that the time step now has a distribution of its own. Recalling the FP derivation, in order to solve the FP equation, we need to solve for $\mathcal{A}(x)$ and $\mathcal{B}(x)$ using the Kolmogorov conditions in equation (12). This necessarily means that we must have a distribution specified for Δx, that is, we must have a distribution function for the spatial step size. Physically this means that each step in the motion of the particle can vary. However, there is no distribution for the time step, Δt. The time step is fixed and acts like a clock that indicates the regular evenly spaced points in time when the particle moves. Now in the CTRW model we will release time from the role of a simple clock and require that Δt also has some statistical distribution and hence variability. In order to proceed further we must therefore specify distribution functions for both Δx and Δt.

Since we have two stochastic variables we can use the Chapman-Kolmogorov equation to relate the joint probability distribution. The Chapman-Kolmogorov equation is given by

$$Q(x,t) = \int d(\Delta x) \int_0^t d(\Delta t) Q(x - \Delta x, t - \Delta t) P(\Delta x, \Delta t)$$

where $Q(x,t)$ is the distribution of jump points and $P(\Delta x, \Delta t)$ is the joint PDF of the two stochastic variables Δx and Δt. To proceed further we must assume some form for $P(\Delta x, \Delta t)$. There are two common models used for specifying the joint PDF called the Waiting model and the Velocity model.

4.2 Waiting Model

The fundamental assumption in the Waiting Model is that the joint PDF can be factored into the product of two separate PDF's each dependent on only one of the stochastic variables, meaning that

$$P(\Delta x, \Delta t) = P(\Delta x)P(\Delta t)$$

If we assume that $P(\Delta x)$ is a standard Gaussian then we can expand $Q(x - \Delta x, t - \Delta t)$ in term of Δx assuming that Δx is small. If we did not have a stochastic dependence on Δt, that is, if t once again played the usual role of a clock, then we would expand Q to second order in Δx which would result in the FP equation (Note that the method mentioned here is equivalent to but not the same as the method in Section 2). However, since we now have Δt as a stochastic variable we expand Q to first order in Δx to get

$$Q_w(x,t) = \int_0^t d(\Delta t) Q(x, t - \Delta t)\Phi_w(\Delta t)$$

where $Q_w(x,t)$ is a relabeling to indicate a jump PDF derived by the Waiting model and $\Phi_w(\Delta t)$ is the probability to wait at least Δt and is given by

$$\Phi_w(\Delta t) = \int_{\Delta t}^{\infty} dt' P(t')$$

To proceed further a distribution must be chosen for $P(t')$, that is, a distribution must be specified for the time step.

4.3 Velocity Model

The Velocity Model proceeds by assuming that the time step is proportional to the spatial step with the constant of proportionality being a finite velocity

$$\frac{\Delta x}{\Delta t} = v \implies \Delta t = \frac{\Delta x}{v}$$

The joint PDF can then be expressed as

$$P(\Delta x, \Delta t) = \delta\left(\Delta t - \frac{|\Delta x|}{v}\right)P(\Delta x)$$

Substituting this joint PDF into the Chapman-Kolmogorv equation gives us

$$Q_v(x,t) = \int_{-vt}^{vt} d(\Delta x) \int_0^t d(\Delta t) Q(x - \Delta x, t - \Delta t)\Phi_v(\Delta x, \Delta t)$$

where $Q_v(x,t)$ is a relabeling to indicate a jump PDF derived by the Velocity model and $\Phi_v(\Delta x, \Delta t)$ is the probability to make a step of at least length $|\Delta z|$ with duration $|\Delta t|$ and is given by

$$\Phi_v(\Delta x, \Delta t) = \frac{1}{2}\delta\big(|\Delta x| - v\Delta t\big) \int_{|\Delta x|}^{\infty} dx' \int_{\Delta t}^{\infty} dt' \delta\left(t' - \frac{|x'|}{v}\right)P(x')$$

Once again to proceed further a distribution must be chosen for $P(x')$, that is, a distribution must be specified for the spatial step.

4.4 CTRW Summary

This completes our short overview of CTRW models. The key assumption is that the distribution now depends on a stochastic time step variable in addition the standard stochastic spatial step variable.

In the case of very small time steps and very small spatial steps, the CTRW methods would reduce to the case of normal diffusion. In other words, anomalous diffusion arises from either long time steps (i.e. sticking) or long spacial steps (i.e. flights). If we have a situation with small spatial steps and Levy distributed times steps (i.e. long wait times; fat tails in PDF) then we would have $\gamma < 1$ which would correspond to subdiffusion. Physically, this would mean that the particle would follow a path where it would stick for long periods of time at certain points. If we have the reverse situation, that is, small time steps and Levy distributed spatial steps (i.e. Levy flights) then we would have $\gamma > 1$ which would correspond to superdiffusion. Physically, this would mean that the particle motion would exhibit a number of very long jumps during its motion. An excellent example of this is the motion of particles in laminar fluid flow in a rotating annulus [1].

Overall the CTRW models are non-local and thus non-Markovian in space time. Thus the history of the particle motion matters as opposed to a Markovian process where the history does not matter. That is, for the FP approach we started by defining a Markov-type chain process in equation (1) which we reproduce below,

$$W(x_3, t_3; x_1, t_1) = \int dx_2 W(x_3, t_3; x_2, t_2) W(x_2, t_2; x_1, t_1)$$

Whereas in the CTRW models we start with a general form of the Chapman-Kolmogorov equation,

$$Q(x, t) = \int d(\Delta x) \int_0^t d(\Delta t) Q(x - \Delta x, t - \Delta t) P(\Delta x, \Delta t)$$

which is non-Markovian due to the space and time distributions.

5 Fractional Kinetic and the FKE

5.1 Derivation of the FKE

We now turn to the discussion of our second method for dealing with anomalous diffusion, namely the Fractional Kinetic method. The FK method is more systematic than the CTRW method but it is slightly more obscure. The FK method and the CTRW method are similar in that both methods change time from a simple clock to a stochastic variable. However, the CTRW method results in non-Markovian equations as functions of $P(\Delta x)$ and $P(\Delta t)$ whereas the FK method results in equations that have fractional derivatives and critical exponents. The culmination of the FK method is the derivation of the Fractional Kinetic Equation (FKE) which is analogous to the FP equation in the FP method.

The derivation of the FKE will closely mirror the FP derivation of the FP equation and as we will see, the FP equation will be a special case of the FKE. We start by defining the transition probability as

$$P(x,t) = W(x,t;x',t') = W(x,x';t-t') = W(x,x';t)$$

where as before the transition probability represents the probability density of a particle to be located at position x at time t if the particle was at position x' at time t' where $t' \leq t$. We once again assume uniformity of time and we neglect x' and t' by assuming that we start from position zero and time zero. We then define $\Delta_t P(x,t)$ to be an infinitesimal shift in $P(x,t)$ along t by Δt. If we had a regular smooth time variable then we could expand $\Delta_t P(x,t)$ as

$$\Delta_t P(x,t) = \frac{\partial P(x,t)}{\partial t}\Delta t + O(\Delta t^2)$$

But since we are now dealing with fractal time, fractional derivatives and fractional exponents, the expansion of $\Delta_t P(x,t)$ becomes

$$\Delta_t^{\beta} P(x,t) = \frac{\partial^{\beta} P(x,t)}{\partial t^{\beta}} + O(\Delta t^{\beta_1}) \quad 0 \leq \beta \leq 1, \quad \beta_1 > \beta \tag{15}$$

We now want to consider an infinitesimal change in $P(x,t)$ due to transitions from other positions, $P(x',t)$, to $P(x,t)$ in the same time interval Δt from above. Since we assume that we have a physical system with finite velocities, for small Δt only the positions of x' that are close to x at time t will be capable of moving to x after a time Δt. We can define $\Delta_x P(x,t)$ to be the infinitesimal shift in $P(x,t)$ due to transitions from other positions. Using fractional exponents once again we have

$$\Delta_x^{\alpha} P(x,t) = \int dy W(x,y;\Delta t)P(y,t) - P(x,t) + O((\Delta t)^{\beta_2}), \qquad \beta_2 > \beta \tag{16}$$

Using equations (15) and (16) we can write an equation that expresses the conservation of particle number

$$\Delta_t^{\beta} P(x,t) = \Delta_x^{\alpha} P(x,t) + O((\Delta t)^{\beta_3}), \qquad \beta_3 = min(\beta_1, \beta_2)$$

Dividing by $(\Delta t)^{\beta}$ and taking the limit at Δt goes to zero gives us

$$\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \Delta_t^{\beta} P(x,t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \Delta_x^{\alpha} P(x,t) \tag{17}$$

Substituting equation (16) into equation (17) gives us

$$\frac{\partial^{\beta} P(x,t)}{\partial t^{\beta}} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \left\{ \int dy W(x,y,\Delta t) P(y,t) - P(x,t) \right\} \tag{18}$$

Note the similarity of equation (18) to equation (4) in the normal FP approach. Now we assume that $W(x, y, \Delta t)$ can be expanded in a similar way to the expansion in equation (5) except now we have to introduce the fractional exponents

$$W(x, y; \Delta t) = \delta(x-y) + A(y, \Delta t)\delta^{\alpha}(x-y) + \frac{1}{2}B(y, \Delta t)\delta^{\alpha_1}(x-y), \quad 0 < \alpha < \alpha_1 \leq 2 \tag{19}$$

Note that if $\alpha = 1$ and $\alpha_1 = 2$ then this reduces to the FP method. In making this expansion we will assume that the functions $A(y, \Delta t)$ and $B(y, \Delta t)$ are independent of $P(x, t)$. Just like in the FP method, we have that $W(x, y; \Delta t)$ represents the local dynamics ($|x - y| \to 0$) and $P(x, t)$ represents the non-local features ($x, t \to \infty$). Thus the assumption of the independence of $A(y, \Delta t)$ and $B(y, \Delta t)$ is akin to the statement that the large time behavior is independent from the local transitions.

Proceeding as before, we look to express $A(y, \Delta t)$ and $B(y, \Delta t)$ as moments of $W(x, y; \Delta t)$. It turns out that $A(y, \Delta t)$ does not have a simple expression but $B(y, \Delta t)$ does. We derive the expression for $B(y, \Delta t)$ by multiplying equation (19) by $|x - y|^{\alpha_1}$ and integrate over x

$$\langle\langle|\Delta x|^{\alpha_1}\rangle\rangle = \int dx |x - y|^{\alpha_1} W(x, y; \Delta t)$$

$$\langle\langle|\Delta x|^{\alpha_1}\rangle\rangle = \int dx |x - y|^{\alpha_1} \left\{\delta(x - y) + A(y, \Delta t)\delta^{\alpha}(x - y) + B(y, \Delta t)\delta^{\alpha_1}(x - y)\right\}$$

Since we have that $\alpha_1 > \alpha$, the first two terms drop out when we use the identities in equation (7)

$$\langle\langle|\Delta x|^{\alpha_1}\rangle\rangle = \alpha_1! B(y, \Delta t)$$

$$\langle\langle|\Delta x|^{\alpha_1}\rangle\rangle = \Gamma(1 + \alpha_1) B(y, \Delta t)$$

To get the more complicated equation for $A(y, \Delta t)$ we integrate equation (19) over y

$$\int dy W(x, y; \Delta t) = \int \delta(x - y) + \int dy A(y, \Delta t)\delta^{\alpha}(x - y) \int dy B(y, \Delta t)\delta^{\alpha_1}(x - y)$$

$$1 = 1 + \int dy \frac{\partial^{\alpha} A(y, \Delta t)}{\partial y^{\alpha}} \delta(x - y) + \int dy \frac{\partial^{\alpha_1} B(y, \Delta t)}{\partial y^{\alpha_1}} \delta(x - y)$$

$$\implies \frac{\partial^{\alpha} A(x, \Delta t)}{\partial(-x)^{\alpha}} + \frac{\partial^{\alpha_1} B(x, \Delta t)}{\partial(-x)^{\alpha_1}} = 0 \tag{20}$$

We then define $\mathcal{A}(x)$ and $\mathcal{B}(x)$ similar to the Kolmogorov conditions in equation (12) of the FP derivation

$$\begin{aligned} \mathcal{A}(x) &= \lim_{\Delta t \to 0} \frac{A(x, \Delta t)}{(\Delta t)^{\beta}} \\ \mathcal{B}(x) &= \lim_{\Delta t \to 0} \frac{B(x, \Delta t)}{(\Delta t)^{\beta}} \end{aligned} \tag{21}$$

Dividing equation (20) by $(\Delta t)^\beta$ and taking the limit as $\Delta t \to 0$ and then using the definitions in equation (21) gives us

$$\frac{\partial^\alpha \mathcal{A}(x)}{\partial(-x)^\alpha} + \frac{\partial^{\alpha_1} \mathcal{B}(x)}{\partial(-x)^{\alpha_1}} = 0 \tag{22}$$

With the definitions in equation (21) and the relation in equation (20) we can now produce the FKE equation. We first rewrite equation (18) as

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta} \left[\int dy \left\{ W(x,y;\Delta t) - \delta(x-y) \right\} P(y,t) \right]$$

Substituting equation (19) into this results in

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta} \left[\int dy \left\{ A(y,\Delta t)\delta^\alpha(x-y) + B(y,\Delta t)\delta^{\alpha_1}(x-y) \right\} P(y,t) \right]$$

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta} \left[\frac{\partial^\alpha}{\partial(-x)^\alpha} \big(A(y,\Delta t) P(x,t) \big) + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}} \big(B(y,\Delta t) P(x,t) \big) \right]$$

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial(-x)^\alpha} \big(\mathcal{A}(x) P(x,t) \big) + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}} \big(\mathcal{B}(x) P(x,t) \big) \tag{23}$$

Equation (23) is the FKE equation and the critical exponents are β, α and α_1. Solving the FKE requires knowledge of the forms of the equations for $\mathcal{A}(x)$ and $\mathcal{B}(x)$.

A simplification of the FKE can be make when $\alpha_1 = \alpha + 1$ and if we use the relation in equation (20). The simplified FKE in this case is

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = -\frac{\partial^\alpha}{\partial(-x)^\alpha} \left(\mathcal{B}(x) \frac{\partial P(x,t)}{\partial x} \right)$$

Note that this equation reduces to the regular FP equation (equation (14)) in the case of $\alpha = 1$, $\beta = 1$ and $\mathcal{B}(x) = \frac{1}{2}D$.

5.2 Special cases of FKE

There are a couple of special cases of the FKE that are worth mentioning. If we assume that $\mathcal{B}(x)$ can be neglected then equation (23) becomes

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial |x|^\alpha} \big(\mathcal{A}(x) P(x,t) \big)$$

Case 1: If $\beta = 1$ and $\alpha = 2$ then we have normal diffusion. Case 2: If $0 < \beta < 1$ and $\alpha = 2$ then we have the equation for fractal Brownian motion

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^2}{\partial |x|^2} \big(\mathcal{A}(x) P(x,t) \big)$$

Case 3: If $\beta = 1$ and $1 < \alpha < 2$ then we have a Levy Process

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^\alpha}{\partial |x|^\alpha} \big(\mathcal{A}(x) P(x,t) \big)$$

5.3 Physics of FKE

When applying the FKE to a physical system we are mostly interested in the moments of $P(x,t)$ since these correspond to the macroscopic observables of the system. In general, the moments are expressed as

$$\langle |x|^\delta \rangle = \int dx |x|^\delta P(x,t) \tag{24}$$

If we assume that $\mathcal{A}(x)$ is constant and $\mathcal{B}(x)$ is negligible in equation (23) then we have

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial |x|^\alpha}\big(\mathcal{A}P(x,t)\big)$$

We then multiply this equation by $|x|^\alpha$

$$\frac{\partial^\beta}{\partial t^\beta}\bigg(|x|^\alpha P(x,t)\bigg) = \mathcal{A}|x|^\alpha \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha}$$

We then integrate over x and use the definition of the moment in equation (24)

$$\frac{\partial^\beta \langle |x|^\alpha \rangle}{\partial t^\beta} = \mathcal{A}\int dx |x|^\alpha \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha}$$

$$\frac{\partial^\beta \langle |x|^\alpha \rangle}{\partial t^\beta} = \mathcal{A}\int dx \bigg(\frac{\partial^\alpha}{\partial |x|^\alpha}|x|^\alpha\bigg) P(x,t)$$

$$\frac{\partial^\beta \langle |x|^\alpha \rangle}{\partial t^\beta} = \alpha! \mathcal{A}\int dx P(x,t)$$

$$\frac{\partial^\beta \langle |x|^\alpha \rangle}{\partial t^\beta} = \Gamma(1+\alpha)\mathcal{A} \tag{25}$$

We then integrate this result over t^β to get

$$\langle |x|^\alpha \rangle = \mathcal{A}\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)} t^\beta \tag{26}$$

For a self-similar solution to the FKE we would expect a relation of the form

$$\langle |x| \rangle \sim t^{\frac{\beta}{\alpha}} = t^{\frac{\mu}{2}}$$

where we have introduced the transport exponent, μ, given by

$$\mu = \frac{2\beta}{\alpha}$$

If we assume that we have finite variances then we can write the variance as

$$\langle x^2 \rangle = t^\mu$$

Thus we see that, in the regime where $\mathcal{A}(x)$ is constant and $\mathcal{B}(x)$ is negligible, the FKE will produce a self-similar result that allows for anomalous diffusion. In addition, we see that the FK method is more general than the FP method since normal diffusion is a special case of equation (26). If $\mu = 1$ then we have $\langle x^2 \rangle = t$ which is the case for normal diffusion (i.e. FP method applies). For $\mu = 1$ we can have $\alpha = 2$ and $\beta = 1$ which agrees with our earlier statements about normal diffusion. The more interesting cases are when $\mu \neq 1$ which is the regime of anomalous diffusion. If $\mu > 1$ then we have super diffusion which physically corresponds to diffusive motion that has very large spatial steps. If $\mu < 1$ then we have subdiffusion which physically corresponds to diffusive motion that has very long wait times. Notice that μ is a function of the critical exponents, α and β, which in turn are determined by the short time evolution of $W(x, y; \Delta t)$ and the Kolmogorov-like conditions in equation (21).

5.4 Applicability of FKE and Conflict with Dynamics

In order to apply the FKE to the description of some physical system, that physical system necessarily puts some restraints on the form and usage of the FKE. There are four primary constraints that are placed on the FKE. (1) If we choose space and time intervals that are infinite then this allows for the possibility of having infinite moments of $P(x, t)$. If we have finite space and time intervals, that is, a space-time window where

$$-\infty < x_{min} < x < x_{max} < \infty \qquad -\infty < t_{min} < t < t_{max} < \infty$$

then we will have finite moments of $P(x, t)$.

(2) Since $P(x, t)$ is a probability we must have that $P(x, t) \geq 0$. This restriction along with the assumption of a system that has infinite space and time intervals (i.e. the possibility of infinite moments) places restrictions on the critical exponents. For example, in the case of the Levy process ($\beta = 1$) these two conditions restrict α to $0 < \alpha \leq 2$. Another example is when $\mathcal{A}(x) = const$ in which case the two conditions restrict the critical exponents to $0 < \beta \leq 1$ and $0 < \alpha \leq 2$.

(3) In the case of finite space-time windows the asymptotics of the systems will be different depending on the window. This means that there will be different sets of critical exponents for different windows.

(4)The definition of the fractional integration and differentiation that we have used in not unique. In our derivation, we have used the Riemann-Liouville form of fractional kinetics but other forms are possible. Using a different form of the fractional derivative will lead to a different structure for the FKE. Thus a physical scenario must specify the type of fractional derivative to use as well as the boundary and initial conditions.

In addition to these four constraints there is an additional constraint that has to do with a result predicted by the FK method that violates the laws of physical dynamics. This violation of dynamics is similar to the infinite velocity problem of the FP method. We assume that $\mathcal{A}$ is constant and consider equation (25)

$$\frac{(\delta x)^\alpha}{(\delta t)^\beta} \sim \mathcal{A}$$

$$v^{\alpha}(\delta t)^{\alpha-\beta} \sim \mathcal{A} = const \tag{27}$$

where v is the velocity. Since $\mu < 2$ we have

$$\alpha - \beta = \alpha\left(1 - \frac{\mu}{2}\right) > 0$$

Thus the exponent of δt in equation (27) is a positive exponent and hence in the limit of $\delta t \to 0$ we must have that $v \to \infty$ which is unphysical. The solution to this problem is that we must allow for a minimum time step, δt_{min}, below which the FKE cannot be applicable. Thus there exists some δt_{min} such that if $\delta t < \delta t_{min}$ then the FKE cannot be applied.

5.5 FK and the Standard Map

The standard map (a.k.a the Chirikov-Taylor map) is a physically useful map since it describes the physics of a magnetic field in a tokamak, a particle interacting with a electromagnetic wave or a periodically kicked rotor. In momentum (p) and position (x) coordinates the standard map is written as

$$p_{n+1} = p_n + Ksin(x_n)$$

$$x_{n+1} = x_n + p_{n+1}$$

where p and x are typically defined on a torus such that $-\pi < p < \pi$ and $-\pi < x < \pi$. For K values that satisfy the Chirikov overlap criterion the motion of the particle will cover all of phase space and it will be chaotic. In the case of normal diffusion the diffusion coefficient for the standard map is given by

$$\mathcal{D} = const = \mathcal{D}_{ql} = \frac{K^2}{2}$$

In addition, the case of regular diffusion is governed by an equation similar in form to equation (14)

$$\frac{\partial P(p,t)}{\partial t} = \frac{1}{2}\mathcal{D}\frac{\partial^2 P(p,t)}{\partial p^2}$$

If we use $\mathcal{D} = \mathcal{D}_{ql}$ then we have that

$$\langle p^2 \rangle = \mathcal{D}t$$

where the second moment of p is linear with respect to time as is expected for normal diffusion. However, simulations show that for certain values of K the second moment of p actually goes as

$$\langle p^2 \rangle = (const) \cdot t^{\mu_p} \quad where\ \mu_p > 1$$

where $\mu_p(K)$ is function of K. This diffusion is superdiffusive and varies depending on the K value. If we plot $\frac{D}{D_{ql}} = \frac{2D}{K^2}$ versus K, as in Figure 1, we see that there are several peaks where the superdiffusion occurs. Normal diffusion occurs when the ratio is equal to one. The presence of anomalous diffusion in the standard map leads to the necessity to use the FK method for analyzing the standard map.

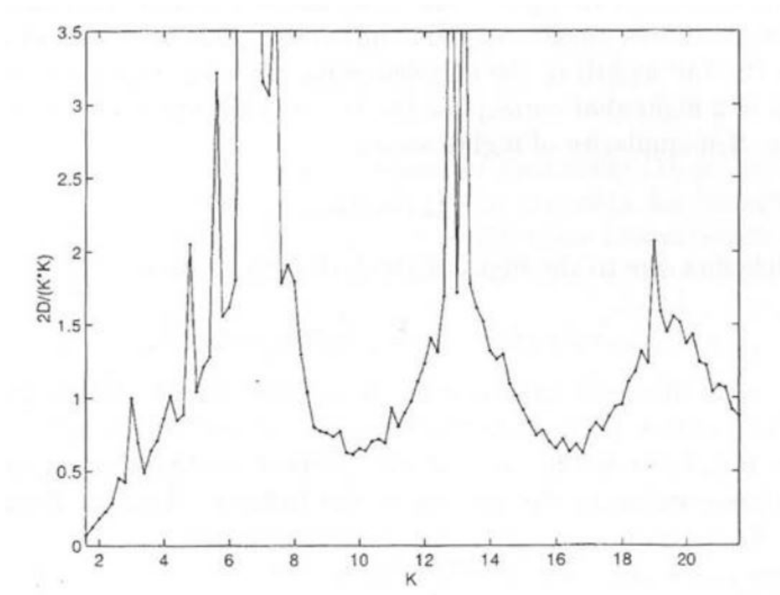

Figure 1: Ratio of diffusion coefficient to normal diffusion coefficient versus K for the Standard Map. Reproduced from [2].

Note that as t increases the peaks get higher and higher. For the standard map we therefore see that there are many sets of critical exponents (α,β) for the system which depend on the K value. In addition, for certain K values the system will undergo normal diffusion while for other values of K the system will undergo super-diffusive levy flights.

6 Comparison of FP and FK methods

In the derivation of the FKE in the previous section we often alluded to the similarities between the FK method and the FP method. Here we summarize that similarity by directly comparing the key equations and parameters in each derivation. The key point is that the FK method is a more general method which in the case of $\beta = 1$, $\alpha = 1$ and $\alpha_1 = 2$ will reduce to the FP method.

Parameter	Fokker-Planck	Fractional Kinetics
Stochastic variable	Δx	Δx, Δt
Role of time	Fixed clock	Variable, PDF
Variance	$\langle \lvert x\rvert^2\rangle \sim t$	$\langle \lvert x\rvert^2\rangle \sim t^\mu$ where $\mu < 2$
Kolmogorov Conditions	Equation (12)	Equation (21)
$A(y, \Delta t)$	$\langle\langle(\Delta y)\rangle\rangle$	No simple form
$B(y, \Delta t)$	$\langle\langle(\Delta y)^2\rangle\rangle$	$\frac{\langle\langle \lvert\Delta x\rvert^{\alpha_1}\rangle\rangle}{\Gamma(1+\alpha_1)}$
Relation between $\mathcal{A}(x)$ and $\mathcal{B}(x)$	Equation (10)	Equation (22)

Perhaps the most important change to the FP method that gives rise to the FK method is encapsulated in rows 1 and 2 where time is given its own statistical distribution and is removed from the role of a fixed clock. This modification is what allows for the description of anomalous diffusion as seen in row 3. The overall similarities between the FK and FP methods are best exhibited by the similarities in the assumptions and key equations such as the Kolmogorov conditions in row 4 and the relation between $\mathcal{A}(x)$ and $\mathcal{B}(x)$ in row 7. For the Kolmogorov conditions we had

$$\begin{array}{cc} FP & FK \\ \lim_{\Delta t\to 0} \frac{1}{\Delta t}\langle\langle \Delta x\rangle\rangle = \mathcal{A}(x) & \mathcal{A}(x) = \lim_{\Delta t\to 0} \frac{A(x,\Delta t)}{(\Delta t)^\beta} \\ \lim_{\Delta t\to 0} \frac{1}{\Delta t}\langle\langle (\Delta x)^2\rangle\rangle = \mathcal{B}(x) & \mathcal{B}(x) = \lim_{\Delta t\to 0} \frac{B(x,\Delta t)}{(\Delta t)^\beta} \end{array}$$

Notice how the key difference between the equations is the usage of a fractional derivative for the FK method. For the relation between $\mathcal{A}(x)$ and $\mathcal{B}(x)$ we had

$$\begin{array}{cc} FP & FK \\ A(y;\Delta t) = \frac{1}{2}\frac{\partial B(y;\Delta t)}{\partial y} & \frac{\partial^\alpha \mathcal{A}(x)}{\partial(-x)^\alpha} + \frac{\partial^{\alpha_1}\mathcal{B}(x)}{\partial(-x)^{\alpha_1}} = 0 \end{array}$$

where once again the key difference is the usage of fractional derivatives in the FK method. The difficulty of using the FK method is best seen in rows 5 and 6 of the table. For the FP method we found that we could easily define $A(y, \Delta t)$ and $B(y, \Delta)$ in terms of the the moments of the PDF which in turn correspond to macroscopic observables. However for the FK method we can only define $B(y, \Delta t)$ in terms of a moment of the PDF. The equation for $A(y, \Delta t)$ is much more difficult to obtain in the FK method which makes the FK method more cumbersome to use than the FP method.

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